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HW Section 6.3 Math 12 Honours Solving Problems Involving Complex Roots

$$r \times e^{i\theta} = r\cos\theta + i \times r\sin\theta \;, \quad \left(re^{i\theta}\right)^n = r^n\left(\cos n\theta + i\sin n\theta\right) \;, \quad \left(re^{i\theta}\right)^{\frac{1}{n}} = r^{\frac{1}{n}}\cos\tfrac{\theta}{n} + r^{\frac{1}{n}}\sin\tfrac{\theta}{n}$$

$$r_1e^{i\theta_1} \times r_2e^{i\theta_1} = r_1 \cdot r_2 \times e^{i(\theta_1+\theta_2)} \;, \quad r_1e^{i\theta_1} \div r_2e^{i\theta_1} = \tfrac{r_1}{r_2} \times e^{i(\theta_1-\theta_2)} \;,$$

1. What are the advantages of writing a complex number in Euler form? $z = r \times e^{i\theta}$ Explain:

Multiplication and division become much faster and easier.

2. How do you find the modulus "r" and argument θ of a complex number when it is in rectangular form: $a+ib$? Explain

$|z| = r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

3. Suppose $z_1 = 5\left(\cos 20° + i\sin 20°\right)$, $z_2 = 11\left(\cos 15° + i\sin 15°\right)$, $z_3 = 4\sqrt{3}\left(\cos 10° + i\sin 10°\right)$, what is the value of $z_1 \times z_2 \times z_3 = ??$

$z_1 \cdot z_2 \cdot z_3 = 5e^{20°i} \cdot 11e^{15°i} \cdot 4\sqrt{3}e^{10°i} = 220\sqrt{3}e^{45i} = \boxed{110\sqrt{6} + 110\sqrt{6}i}$

4. Suppose $5\left(\cos 20° + i\sin 20°\right)^5 = z^3$, what is the modulus and argument of z ?

$r = 5$ $\theta = 20°$

5. Is the following equation true? Explain: $\frac{1}{z} = \bar{z}$

$\underline{\underline{\text{no}}}$ $\frac{1}{a+bi} \neq a-bi \Rightarrow \frac{a-bi}{a^2+b^2} \neq a-bi$

6. Are the following complex numbers equivalent? $z_1 = 2 \times e^{\frac{2\pi}{3}i}$ and $z_2 = 2 \times e^{\frac{-\pi}{3}i}$ Explain:

NO, $z_1 = -z_2$

7. Suppose $z^{12} = 1$, how many solutions are there? How many of the solutions are real? How many of them are complex? Explain:

There are 12 solutions.
1 real, 11 complex

8. Given that $z^4 = 2\sqrt{3} + 2i$, how many solutions are there? How would you find the solutions? Explain:

There are 4 solutions equally spaced around the unit circle. By finding one, we can find the rest by adding 90°.

9. Given the equation $z^{24} = 1$ with 24 solutions for $0° \leq \theta \leq 360°$, if we take the sum of the real component of all 24 solutions, what would it be equal to? If we take the sum of all the imaginary component of all 24 solutions, what would it be equal to? Explain:

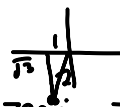
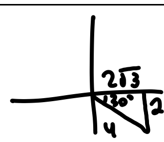
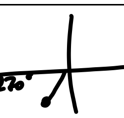
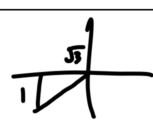
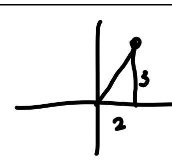
They would both equal to zero, by way of vieta sums:

$z^{24} = 1 \Rightarrow -\frac{b}{a} = \frac{-0}{1} =$ sum of roots

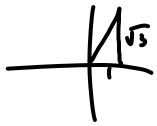
10. Convert each of the following complex numbers into Euler form:

a) $2\sqrt{3}-2i$ $=4e^{-30^\circ i}$	b) $5i-5\sqrt{3}$ $=10e^{150^\circ i}$	c) $6-3i$ $=3\sqrt{5}e^{-45^\circ i}$
d) $10-10i$ $=10\sqrt{2}e^{-45^\circ i}$	e) $-13i$ $=-13e^{-90^\circ}$	f) $4(3\sqrt{2}i-5)$ $=4\sqrt{43}e^{139.7^\circ i}$
g) $\frac{1-\sqrt{3}i}{2\sqrt{3}+2i}=\frac{1-\sqrt{3}i}{2(\sqrt{3}+i)}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}=\frac{-4i}{8}$ $=-\frac{1}{2}i=\frac{1}{2}e^{-90^\circ i}$	h) $\frac{13-2i}{4+3i}\times\frac{4-3i}{4-3i}=\frac{46-47i}{25}$ $=\frac{\sqrt{173}}{5}e^{134.4^\circ i}$	i) $\frac{6-7i}{3-5i}\times\frac{3+5i}{3+5i}=\frac{53+9i}{34}$ $=\frac{\sqrt{10}}{2}e^{9.64^\circ i}$

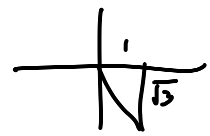
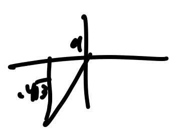
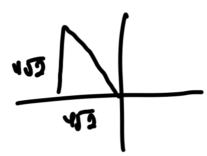
11. Evaluate each of the following using De Moivre's theorem and express in rectangular form:

i) $(\sqrt{3}+i)^5$ $=(2e^{30^\circ i})^5=32e^{150^\circ i}=32(\cos 150^\circ+i\sin 150^\circ)$ $=\boxed{-16\sqrt{3}+16i}$	ii) $(-1-i\sqrt{3})^6$  $=(2e^{-120^\circ i})^6=64e^{-720^\circ i}=\boxed{64}$
iii) $(2\sqrt{3}-2i)^{12}$ $=(4e^{-30^\circ i})^{12}=4^{12}e^{-360^\circ i}=\boxed{4^{12}}$ 	iv) $(-\sqrt{2}-\sqrt{2}i)^{10}$ $=(2e^{-135^\circ i})^{10}=1024e^{-1350^\circ i}=1024e^{-270^\circ}$  $=\boxed{1024i}$
v) $(-\sqrt{3}-i)^5$  $=(2e^{-150^\circ i})^5=32e^{-750^\circ i}=32e^{-30^\circ i}$ $=32[\cos(-30^\circ)+i\sin(-30)]$ $=\boxed{16\sqrt{3}-16i}$	vi) $(2+3i)^4$  $=(\sqrt{13}e^{56.31^\circ i})^4$ $=169e^{225.24^\circ i}=\boxed{-119-120i}$

12. Solve for z, with $0° \le \theta \le 360°$:

a) $z^3 = -1$ $z_1 = -1$ $z_2 = -e^{120^\circ i} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ $z_3 = -e^{-120^\circ i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	b) $z^4 = 1 + i$ $z^4 = \sqrt{2}e^{45i}$ $z_1 = \sqrt[4]{2}e^{\frac{45}{4}i}$ $\boxed{z = \sqrt[8]{2}\, e^{\frac{45}{4}i \pm 90^\circ i}}$
c) $z^3 = 1 + \sqrt{3}i$  $z^3 = 2e^{60^\circ i}$ $z_1 = \sqrt[3]{2}\, e^{20^\circ i}$ $z_2 = \sqrt[3]{2}\, e^{140^\circ i}$ $z_3 = \sqrt[3]{2}\, e^{260^\circ i}$	d) $z^3 + 8 = 0$ $z^3 = -8$ $z_1 = -2$ $z_2 = -2e^{120^\circ i}$ $z_3 = -2e^{240^\circ i}$

13. Find the complex roots for the following expressions:

i) Fifth roots of $3+3i$: $z^5 = 3+3i$ $z^5 = 3\sqrt{2}e^{45^\circ i}$ $z_1 = \sqrt{18}\,e^{5i}$ $z_2 = \sqrt{18}\,e^{77^\circ i}$ $z_3 = \sqrt{18}\,e^{149^\circ i}$ $z_4 = \sqrt{18}\,e^{221^\circ i}$ $z_5 = \sqrt{18}\,e^{293i}$	ii) sixth roots $1-\sqrt{3}i$ $z^6 = 1-\sqrt{3}i$  $z^6 = 2e^{-60^\circ i}$ $z_1 = \sqrt[6]{2}\, e^{-10^\circ i}$ $z_2 = \sqrt[6]{2}\, e^{50^\circ i}$ $z_3 = \sqrt[6]{2}\, e^{110^\circ i}$ $z_4 = \sqrt[6]{2}\, e^{170^\circ i}$ $z_5 = \sqrt[6]{2}\, e^{230^\circ i}$ $z_6 = \sqrt[6]{2}\, e^{290^\circ i}$
iii)Fourth Roots of $-4\sqrt{3}-4i$  $z^4 = -4\sqrt{3} - 4i = 8e^{-120^\circ i}$ $z_1 = \sqrt[4]{8}\, e^{-30^\circ i}$ $z_2 = \sqrt[4]{8}\, e^{60^\circ i}$ $z_3 = \sqrt[4]{8}\, e^{150^\circ i}$ $z_4 = \sqrt[4]{8}\, e^{240^\circ i}$	iv)Cube Roots of $-4\sqrt{2}+4i\sqrt{2}$  $z^3 = -4\sqrt{2} + 4i\sqrt{2}$ $z^3 = 8e^{135^\circ i}$ $z_1 = 2e^{45^\circ i}$ $z_2 = 2e^{165^\circ i}$ $z_3 = 2e^{285^\circ i}$

14. Solve for "z": $z = \sqrt[4]{i}$

$z_1 = \sqrt[4]{e^{90°i}} = e^{22.5°i}$

$z_2 = e^{22.5i + 90°} = e^{112.5°i}$

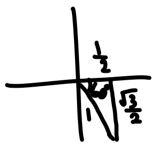
$z_3 = e^{202.5°i}$

$z_4 = e^{292.5°i}$

15. Using De Moivre's theorem, find the values of "a" and "b": $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{20} = a + ib$

$\frac{1}{2} - \frac{i\sqrt{3}}{2} = e^{-60°i}$ $\quad \left(e^{-60°i}\right)^{20} = e^{-1200°i} = e^{-120°i} = \cos(-120°) + \sin(-120°)i$

$= \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$



16. Evaluate: i) $\frac{(1-i\sqrt{3})^3}{(-2+2i)^4}$

$= \frac{\left(2e^{60°i}\right)^3}{\left(2\sqrt{2}e^{135°i}\right)^4} = \frac{8e^{180°i}}{64e^{180i}} = \frac{8}{64} = \boxed{\frac{1}{8}}$

b) Evaluate: $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^9$

$= \left(e^{30°i}\right)^9 = e^{270°i} = \boxed{-i}$

17. Prove that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{(\cos\theta + i\sin\theta) - [\cos(-\theta) + i\sin(-\theta)]}{2i} = \frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{2i}$

$= \frac{\cancel{2i}\sin\theta}{\cancel{2i}} = \boxed{\sin\theta}$

18. The points (0,0) , (a,11) , and (b,37) are the vertices of an equilateral triangle. Find the value of "ab". (1994 Aime)

z_1 $\quad z_2$

(a,11) and (b,37) must be separated by angle of 60° from (0,0). Their distance from (0,0) must also be 0. $\Rightarrow \sqrt{a^2+11^2} = \sqrt{b^2+37^2}$

$a^2 + 11^2 = b^2 + 37^2$

$a + 11i = re^{i\theta}$ $\quad a = r\cos\theta \quad 11 = r\sin\theta$

$b + 37i = re^{i(\theta+60°)}$ $\quad b = r\cos(\theta+60) = r\left(\cos\theta \cdot \frac{1}{2} - \sin\theta \cdot \frac{\sqrt{3}}{2}\right) \Rightarrow b = \frac{a}{2} - \frac{11\sqrt{3}}{2} = \frac{a - 11\sqrt{3}}{2}$

$a^2 + 11^2 = \left(\frac{a - 11\sqrt{3}}{2}\right)^2 + 37^2$

$a = 21\sqrt{3}$

$b = 5\sqrt{3}$

$\boxed{ab = 315}$

19. The solutions to the equation $(z+6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are "A", "B", and "C". What is the least possible area of triangle ABC?

$(z+6)^8 = 81 \xrightarrow{\text{H.S. GR}} z^8 = 81$

roots of unity

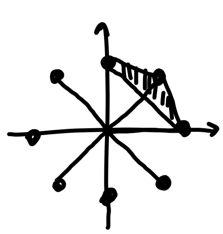
$\Longrightarrow z_1 = \sqrt{3}$

$z_2 = \sqrt{3}e^{i45°} = \frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}i$

$z_3 = \sqrt{3}e^{i90°} = \sqrt{3}i$

Shoelace:

	$\sqrt{3}$,	0	
0	$\frac{\sqrt{6}}{2}$,	$\frac{\sqrt{6}}{2}$	$\frac{3\sqrt{2}}{2}$
0	0,	$\sqrt{3}$	$\frac{3\sqrt{2}}{2}$
$\frac{3}{3}$	$\sqrt{3}$,	0	$\frac{0}{3\sqrt{2}}$

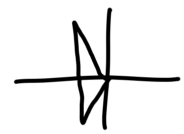


$\boxed{\text{Area} = \frac{3\sqrt{2} - 3}{2}}$

20. If we know that $|z| = 1$ and given that "K' is a real number, what is the maximum value of $|z^2 + Kz + 1|$?

Note: $|z|$ refers to the modulus of the complex variable z.

If K has to be a real numbers, then it can be infinitely large while
$z = 1$ ...



21. If $z + \frac{1}{z} = 2\cos 3°$, then what is the value $z^{2000} + \frac{1}{z^{2000}} = ?$ (AIME 2000)

$$\frac{z^2+1}{z} = 2\cos 3°$$

$$z^2 - 2z\cos 3° + 1 = 0 \Rightarrow z = \frac{2\cos 3° \pm \sqrt{4\cos^2 3° - 4}}{2}$$

$$\Rightarrow z = \cos 3° \pm i\sin 3° = e^{3i}$$

$$z^{2000} + \frac{1}{z^{2000}} = e^{3i\cdot 2000} + e^{-3i\cdot 2000}$$

$$= e^{240i} + e^{-240i}$$

$$= e^{240i} + e^{120i} = 2\mathrm{Re}(e^{120i}) = \boxed{-1}$$

Problem 4

Let $w = \frac{\sqrt{3}+i}{2}$ and $z = \frac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

$$\omega = \frac{\sqrt{3}+i}{2} = e^{30°i}$$

$$z = \frac{-1+i\sqrt{3}}{2} = e^{120°i}$$

$$i\left(e^{30°i}\right)^r = \left(e^{120°i}\right)^s$$

$$e^{90°i + 30°r} = e^{120i\cdot s}$$

r	s	# possibilities
1	1, 4, 7, ⋯ , 100	34
2, 3, 4	∅	
5	2, 5, 8, ⋯ , 98	33
6, 7, 8	∅	
9	3, 6, 9, ⋯ , 99	33
⋮	⋮	⋮
97	1, 4, 7, ⋯ , 100	34

$$\text{Total} = (33+34+33)\cdot 8 + 34 = \boxed{834}$$